Electric and Magnetic Potentials of Uniformly Accelerated Charge Distributions

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Jefimenko gave recently [1] new expressions for the electric and magnetic potentials of uniformly moving, time-independent charge distributions. We discuss these potentials for uniformly accelerated distributions. As Jefimenko did, we implement two procedures either converting directly retarded into present position integrals or using relativistic transformations for a stationary charge in an instantaneous comoving inertial frame (Frenet-Serret tetrad). We discuss why, at the difference of what happens for uniform motions [1], both procedures provide different expressions for potentials.

1. Introduction

As a contribution to the relations between retardation and relativity [2]–[7], Jefimenko gave recently [1] new integrals for electric and magnetic potentials of uniformly moving, time-independent charge distributions. In contrast to the familiar retarded integrals, these new integrals that following Jefimenko we name "present position" integrals, express the potentials in terms of the position occupied by the charge distribution at the moment for which the potentials are to be determined. We discuss here what happens to these integrals for charge distributions moving with constant acceleration (hyperbolic motion).

Two procedures can be implemented to get present position integrals. One may start from retarded integrals and use the equations of motion to translate the integrand to the position occupied by the charge at the time of observation. One may also start from the Coulomb potential of a stationary charge in a comoving inertial frame and apply the relativistic transformations between inertial frames. For uniformly moving charges, the comoving frame is inertial and both procedures give the same expression [1]. For uniformly accelerated charges the situation is different and requires some discussion.

Minkowski [8] was the first to recognize the interest in special relativity of hyperbolic motions later discussed more fully by Born [9] and Sommerfeld [10] with an application to a point charge describing such a motion [9], [11] (see Pauli [12] for a more complete discussion and [13]–[15] for recent works on electromagnetism in uniformly accelerated media). As suggested by De Donder [16] there exists in uniformly accelerated systems an instantaneous comoving inertial frame S [17]–[19] in which the observer performs his measurements so that between S and the laboratory inertial frame L one may use the same transformations as between L and a uniformly moving frame R, except that the velocity between L and S is a function of space and time. And it is known [13], [20], [21] that for a system moving along a curve Γ in space-time, as it is the case for hyperbolic motions, the inertial frame S is provided by the Frenet-Serret tetrad [22] that we shall define later.

There are many ways to describe an hyperbolic motion [18] but to get the analogue of a classical charge distribution, we need equations of the motion such that the velocity does not depend on the position inside the charge distribution. And essentially two different descriptions are used in the literature: the Pauli-Rindler picture [12], [23], [24],

$$x=x'+b(\cosh pt'-1), \quad y=y', \quad z=z'$$
$$p(t+t_0)=\sinh pt', \quad p=a/c, \quad b=c/p=c^2/a \quad (1)$$

and the Möller picture [25], [26], [27],

$$x+b=(x'+b)\cosh pt', \quad y=y', \quad z=z'$$
$$c(t+t_0)=(x'+b)\sinh pt'. \quad (2)$$

In these relations (x, t) and (x', t') denote respectively the space-time coordinates in the Minkowski space L and in the coaccelerating frame K, t' is the proper time, a is a uniform acceleration, c the velocity of light and t_0 the time when the hyperbolic motion starts. Both systems have been adjusted so that $t'=0$ at

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$t=-t_0$ with the relative velocity zero and $x=x'$, $y=y'$, $z=z'$. The velocity $v(t, t_0)$ that we assume in the direction $x<0$ is

$$v=c \tanh pt', \quad \beta=\tanh pt',$$
$$\gamma=(1-\beta^2)^{-1/2}=\cosh pt'. \tag{3}$$

v, β, γ are shorthand notations and we have from (1)

$$\beta(t, t_0)=p(t+t_0)\,[1+p^2(t+t_0)]^{-1/2},$$
$$\gamma(t, t_0)=[1+p^2(t+t_0)^2]^{1/2} \tag{4a}$$

and from (2)

$$\beta(x'; t, t_0)=p(t+t_0)\,[(1+x'/b)^2+p^2(t+t_0)^2]^{-1/2}, \tag{4b}$$
$$\gamma(x'; t, t_0)=(1+x'/b)^{-1}\,[(1+x'/b)^2+p^2(t+t_0)^2]^{1/2}.$$

In contrast to (4a), the equations (4b) show that for the Möller hyperbolic motion the velocity changes from point to point in the comoving frame in opposition to the condition previously imposed on the charge distribution which would appear as nonrigid with respect to the laboratory frame. So from now on we only work with (1) that we write after elimination of t'

$$x=x'-b+b[1+p^2(t+t_0)^2]^{1/2} \tag{5}$$

which reduces for $p(t+t_0)$ small enough to the newtonian equation

$$x=x'+a(t+t_0)^2/2. \tag{5a}$$

We assume that during a measurement process one may neglect the variation of acceleration so that the measurements are equivalent for the accelerated observer and for the inertial observer having the same velocity.

2. Retarded and Present Position Integrals for Potentials

We now interpret (x', t') and (x, t) as the space-time coordinates of the radiation source at the retarded time and at the time of observation, respectively. We assume (see Fig. 1) that a charge distribution is at rest at the time $-t_0$ when it is put in motion with the constant acceleration a according to (5). Then, using the notation

$$d\phi(x', t')=\varrho(x', t')\,|x-x'|^{-1}\,d^3x', \tag{6}$$

the electric potential $\phi(x, t)$ for a time variable charge distribution is given for $t\leq -t_0$ by the Coulomb law

$$\phi(x, t)=(4\pi\varepsilon_0)^{-1}\int d\phi(x', t), \quad t\leq -t_0, \tag{7}$$

and for $t>-t_0$ by the retarded integral

$$\phi(x, t)=(4\pi\varepsilon_0)^{-1}\int [d\phi(x', t')], \quad t>-t_0 \tag{8}$$

that, for a time-independent charge distribution reduces to

$$\phi(x, t)=(4\pi\varepsilon_0)^{-1}\int [d\phi(x')], \quad t>-t_0. \tag{8a}$$

Generalizing the Jefimenko result [1] we shall show how this last integral can be transformed into the present position integral

$$\phi(x, t)=(4\pi\varepsilon_0)^{-1}\int \{d\phi(x', t')\}, \quad t>-t_0. \tag{8b}$$

In these integrals, $[d\phi]$ and $\{d\phi\}$ are the expressions of $d\phi$ at the times $t_r=t-|x-x'|/c$ and t, respectively. Now the current density for a time-independent charge moving with the velocity $v(t)$ is $j(x, t)=\varrho(x)\,v(t)$, so with

$$dA(x', t')=j(x', t')\,|x-x'|^{-1}\,d^3x' \tag{9}$$

the retarded integral for the magnetic vector potential is

$$A(x, t)=(4\pi\varepsilon_0 c^2)^{-1}\int [dA(x', t')], \tag{10a}$$

and we shall give for a uniformly accelerated motion the expression of the present position integral

$$A(x, t)=(4\pi\varepsilon_0 c^2)^{-1}\int \{dA(x', t')\}. \tag{10b}$$

2.1 Scalar Potential

We first explicit (8a, b) for a time-independent charge $\varrho(x)$ with the arbitrary velocity $v(t)$ in the x-direction (an excellent discussion for a moving point charge is given in [28]). For simplification, we assume that the observer at the origin 0 of coordinates makes his measurements at $t=0$ (the location and the time of observation have no effect on the result but assuming $x=t=0$ simplifies calculations). Then, the retarded integral (8a) becomes

$$\phi(0, 0)=(4\pi\varepsilon_0)^{-1}\int r'^{-1}\,\varrho(x')\,d^3x', \quad t>-t_0. \tag{11}$$

x' is the position of the radiation source at the time $t_r=-r'/c$, $r'^2=x'^2+y'^2+z'^2$ and x its position at $t=0$. So the relation between the abscissae x and x' is

$$x=x'+\int_{t_r}^{0} v(t)\,dt,$$
$$=x'+X(0)-X(t_r). \tag{12}$$

From (12) we get

$$dx=(1-x'\,\dot{X}/c\,r')\,dx', \quad \dot{X}=\partial X/\partial t_r, \tag{13}$$

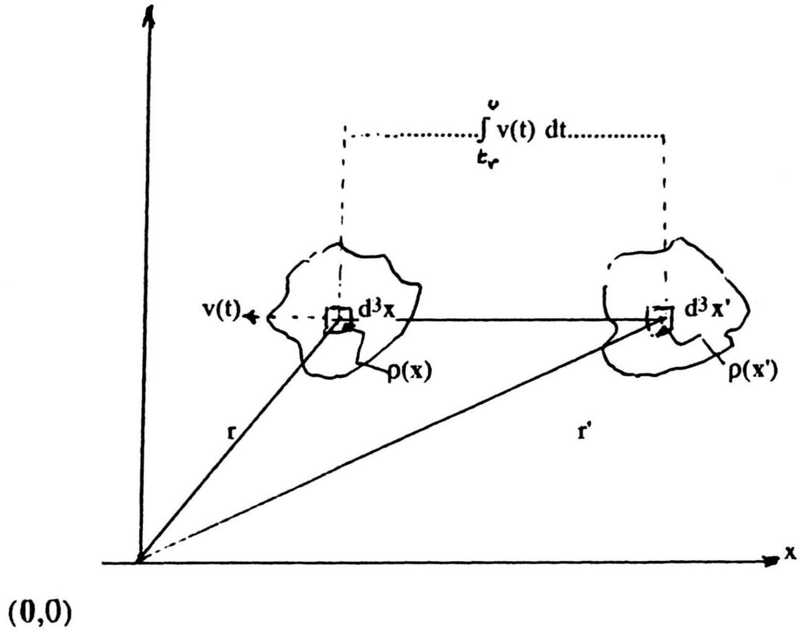

Fig. 1. Uniformly accelerated charge distribution.

Substituting (13) into (11) gives the present position integral

$$\phi(0, 0)=(4\pi\varepsilon_0)^{-1}\int(r'-x'\,X/c)^{-1}\varrho(x)\,d^3x, \qquad (14)$$

in which x' and r' are expressed in terms of x and $r=(x^2+y^2+z^2)^{1/2}$ with the help of (12). For a uniform motion

$$X=v, \quad r'-x'\,X/c=[x^2+\gamma^{-2}(y^2+z^2)]^{1/2} \qquad (15)$$

and (14) becomes the Jefimenko expression [1].

For the hyperbolic motion of a charge distribution starting to move at $t=-t_0$ with the velocity $v(t)=p(t+t_0)\,[1+p^2(t+t_0)^2]^{-1/2}$ the relation (12) becomes

$$x+x'+(b^2+c^2\,t_0^2)^{1/2}-[b^2+(r'-c\,t_0)^2]^{1/2}, \qquad (16)$$

a result that we could have obtained directly from (5) by making successively $t=-r'/c$ and $t=0$ and substracting the two corresponding expressions.

Using (16) to get x' in terms of x gives very intricate formulae. But the parameter b is generally very large, for instance [20] b is about 10^{16} m, which is approximately one light year for $a=9$ ms^{-2}. Then, assuming $c\,t_0 \ll b$ and neglecting the terms in b^{-3} and higher, we get to the order $O(b^{-3})$

$$x=x'-(r'^2-2\,c\,t_0\,r')/2\,b+O(b^{-3}). \qquad (17)$$

We also get this equation by assuming a small enough acceleration so that on the intervall $[-t_0\;0)$ the hyperbolic motion can be described by the newtonian equation (5a).

From now on, all calculations are made to the order $O(b^{-3})$ and this symbol is suppressed. From (17) we get

$$dx=(1-x'(r'-c\,t_0)/b\,r')\,dx', \qquad (18)$$

and a simple calculation in which $r^2=x^2+y^2+z^2$ gives

$$x'=x+b^{-1}f(x)+b^{-2}g(x), \qquad (19)$$

$$f(x)=r(r-2\,c\,t_0)/2, \quad g(x)=x(r-c\,t_0)\,f(x)/r, \qquad (19\text{a})$$

so that $(s^2=y^2+z^2)$

$$r'=r+b^{-1}r^{-1}x\,f(x)+b^{-2}r^{-1}f(x)\cdot[s^2\,r^{-2}f(x)+2\,x^2\,r^{-1}(r-c\,t_0)]/2. \qquad (20)$$

From (18), (19), (20) we get

$$dx'/r'=dx\,b[b\,r'-x'(r'-c\,t_0)]^{-1} = dx/r\,B(x), \quad B(x)=1-x/2\,b+(r-c\,t_0)\cdot[s^2\,r^{-2}(r-c\,t_0)-4\,r]/8\,b^2. \qquad (21)$$

Substituting (21) into (11) gives to the order $O(b^{-3})$ the present position scalar potential for the hyperbolic motion (1)

$$\phi(0, 0)=(4\pi\varepsilon_0)^{-1}\int r^{-1}B^{-1}(x)\,\varrho(x)\,d^3x. \qquad (22)$$

As easily seen, the main difficulty for a nonuniform motion is to get from (12) x' in terms of x.

2.2 *Vector Potential*

With the same assumptions as in the previous section, the retarded current density for a time-independent charge distribution moving according to (1) is

$$j_x(x', t') = \varrho(x')\, v(t_r, t_0), \quad j_y(x', t') = j_z(x', t') = 0 \tag{23}$$

with according to (4a) and since $t_r = -r'/c$

$$v(t_r, t_0) = p(t_0 - r'/c)\,[1 + p^2 (t_0 - r'/c)^2]^{-1/2}. \tag{23a}$$

So for the retarded potential we have $A_y = A_z = 0$ and

$$A_x(0, 0) = (4\pi \varepsilon_0 c^2)^{-1} \int r'^{-1} \varrho(x')\, v(t_r, t_0)\, \mathrm{d}^3 x'. \tag{24}$$

Then, using (18), (19), (20) and the relation $v(0, t_0) = p\, t_0 (1 + p^2 t_0^2)^{-1/2}$ we get to the order $0(b^{-3})$ the present position integral for the component A_x of the vector potential

$$A_x(0, 0) = (4\pi \varepsilon_0 c^2)\, v(0, t_0) \int r^{-1} B^{-1}(x)\, \varrho(x)\, \mathrm{d}^3 x. \tag{25}$$

Such a result could not have been obtained for the Möller hyperbolic motion (2) since in this case the velocity depends on x'.

3. Relativistic Transformation of Potentials

3.1 *Frenet-Serret Tetrad*

As said in the introduction, for a system moving along a curve Γ in space-time the instantaneous comoving inertial frame S is provided by the Frenet-Serret tetrad made of four orthogonal unit vectors $\boldsymbol{e}_{(\mu)}$ at each point of the curve. The four velocity u^μ is taken as time-like unit vector $\boldsymbol{u}^\mu \boldsymbol{e}_{(0)}$, and the three space-like vectors $\boldsymbol{e}_{(k)}$ are obtained from the relations

[24] $$\partial/\partial s\, \boldsymbol{e}_{(0)} = a\, \boldsymbol{e}_{(1)}, \quad \partial/\partial s\, \boldsymbol{e}_{(1)} = b\, \boldsymbol{e}_{(2)} + a\, \boldsymbol{e}_{(1)}, \tag{26}$$
$$\partial/\partial s\, \boldsymbol{e}_{(2)} = c\, \boldsymbol{e}_{(3)} = c\, \boldsymbol{e}_{(1)} - b\, \boldsymbol{e}_{(1)}, \quad \partial/\partial s\, \boldsymbol{e}_{(3)} = -c\, \boldsymbol{e}_{(2)}.$$

The coefficients a, b, c, are nonnegative and $\partial/\partial s$ is the absolute derivative. For a four vector $\boldsymbol{A}^\mu$ we have

$$\partial/\partial s\, \boldsymbol{A}^\mu = u^\nu\, \partial_\nu\, \boldsymbol{A}^\mu + \Gamma^\mu_{\alpha\beta}\, \boldsymbol{A}^\alpha\, u^\beta \cdot \tag{27}$$

$\partial_\mu = \partial/\partial x^\mu$ and the parameters $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. The greek (resp.latin) indices take the values 0, 1, 2, 3 (resp. 1, 2, 3), and we use the summation convention. With x denoting coordinates and $\boldsymbol{e}$ unit vectors we use the following notations respectively in the laboratory frame L, the comoving frame K and the instantaneous comoving inertial frame S

$$\{x^\mu(ct, x, y, z); \boldsymbol{e}_\mu\}_L, \quad \{x'^\mu(ct', x', y', z'); \boldsymbol{e}'_\mu\}_K,$$
$$\{x^{(\mu)}(cT, X, Y, Z); \boldsymbol{e}_{(\mu)}\}_S, \tag{28}$$

Let us now define the Frenet-Serret tetrad for the hyperbolic motion (1). Using cartesian coordinates, the metric of the Minkowski space-time in the laboratory frame has the simple form $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x_i\,\mathrm{d}x^i$ so that the Christoffel symbols are zero. According to (3), the three velocity of the hyperbolic motion is $v = c \tanh pt'$ so that the corresponding four velocity is

$$u^\mu = (\cosh pt', \sinh pt', 0, 0)$$

and consequently

$$\boldsymbol{e}_{(0)} = \cosh pt'\, \boldsymbol{e}_0 + \sinh pt'\, \boldsymbol{e}_1\,. \tag{29a}$$

Using (26), (27) and the relation $\partial_{t'} = \cosh pt'\, \partial_t + \sinh pt'\, c^{-1}\, \partial_x$ we get from (29a)

$$\boldsymbol{e}_{(1)} = \sinh pt'\, \boldsymbol{e}_0 + \cosh pt'\, \boldsymbol{e}_1, \quad \boldsymbol{e}_{(2)} = \boldsymbol{e}_2, \quad \boldsymbol{e}_{(3)} = \boldsymbol{e}_3\,. \tag{29b}$$

So the instantaneous Lorentz transform between the laboratory inertial frame L and the comoving inertial frame S (Frenet-Serret tetrad) is

$$\Lambda^\nu_{(\mu)} = \begin{vmatrix} \cosh pt' & \sinh pt' & 0 \\ \sinh pt' & \cosh pt' & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{30}$$

that is with the notations (28)

$$c\,\mathrm{d}t = \cosh pt'\, c\,\mathrm{d}T + \sinh pt'\,\mathrm{d}X,$$
$$\mathrm{d}x = \sinh pt'\, c\,\mathrm{d}t + \cosh pt'\,\mathrm{d}X\,. \tag{31}$$

Let us now consider the two four vectors $\boldsymbol{j}$, $\boldsymbol{A}$ having respectively ϱ and ϕ as their zero component while their other components correspond to the current density and to the magnetic potential. From the relations $j^\mu = \Lambda^\mu_{(\nu)}\, j^{(\nu)}$, $A^\mu = \Lambda^\mu_{(\nu)}\, A^{(\nu)}$ we get for a charge at rest in the Frenet-Serret tetrad ($j^{(k)} = A^{(k)} = 0$

$$\varrho^0(x) = \cosh pt'\, \varrho^{(0)}(X),$$
$$j^1(x = \sinh pt'\, \varrho^{(0)}(X), \quad j^2 = j^3 = 0,$$
$$\phi^0(x) = \cosh pt'\, \phi^{(0)}(X),$$
$$A^1(x) = \sinh pt'\, \phi^{(0)}(X), \quad A^2 = A^3 = 0. \tag{32}$$

Let us now explicit the relations between four vectors in the two comoving frames K and S. Starting from the relations between the unit vectors in the L and K frames

$$\boldsymbol{e}'_0 = \cosh pt'\, \boldsymbol{e}_0 + \sinh pt'\, \boldsymbol{e}_1, \quad \boldsymbol{e}'_j = \boldsymbol{e}_j \tag{33}$$

we get from (29)

$$e_{(0)} = e'_0, \quad e_{(2)} = e'_2, \quad e_{(3)} = e'_3,$$
$$e_{(1)} = \tanh pt'\, e'_0 + (\cosh pt')^{-1}\, e'_1 \tag{34}$$

so that

$$c\, dt' = c\, dT + \tanh pt'\, dX,$$
$$dx' = (\cosh pt')^{-1}\, dX. \tag{35}$$

Eliminating dX, dX in (35) with the help of (31) leads to an integrable system of partial differential equations supplying the equation (5). A charge distribution at rest in the Frenet-Serret tetrad is also at rest in the K frame. And in this case the charge and current densities as well as potentials transform according to (34) as

$$\varrho'^0(x') = \varrho^{(0)}(X), \quad j'^k(x') = j^{(k)}(X),$$
$$\phi'^0(x') = \phi^{(0)}(X), \quad A'^k(x') = A^{(k)}(X). \tag{36}$$

Maxwell's equations in the comoving frame K are discussed in [15].

3.2 Potentials in the Frenet-Serret Tetrad

In the Frenet-Serret tetrad S in which the charge element $d\varrho = \varrho^{(0)}(X)\, d^3 X$ is at rest at the point X the electric potential $\phi^{(0)}(X_p)$ at a point X_p in S is given by the Coulomb law

$$\phi^{(0)}(X_p) = (4\pi\varepsilon_0)^{-1} \int |X - X_p|^{-1}\, \varrho^{(0)}(X)\, d^3 X. \tag{37}$$

Now we first remark that for $t=0$ we get from (4a) and (31)

$$\cosh pt' = (1 + p^2 t_0^2)^{1/2} = \gamma, \quad dX = d(\gamma x), \tag{38}$$

and that the point x'_p in the K-frame corresponding to the origin $x = t = 0$ of the L-frame is according to (5)

$$x'_p = b(1-\gamma), \quad y'_p = z'_p = 0, \tag{39}$$

so that integrating the second equation (35) gives the point X_p in the Frenet-Serret tetrad

$$X_p = b\gamma(1-\gamma), \quad Y_p = Z_p = 0. \tag{39a}$$

Then, using (32), (38) and taking into account (39a), the integral (37) becomes in the laboratory frame

$$\gamma^{-1}\phi^0(0,0)$$
$$= (4\pi\varepsilon_0)^{-1} \int [\gamma^2 x^2 + y^2 + z^2]^{-1/2}\, \gamma^{-1}\, \varrho^0(x)\, d^3(\gamma x),$$

so

$$\phi^0(0,0) = (4\pi\varepsilon_0)^{-1} \int [x^2 + \gamma^2(y^2+z^2)]^{-1/2}\, \varrho^0(x)\, d^3 x. \tag{40}$$

This expression is formally the same as for a uniformly moving charge [1] but here γ is an instantaneous factor depending on the time t_0 when the charge distribution is put in motion and on the time of observation.

For the vector potential we get from (32) $A^1 = \tanh pt'\, \phi^0$, $A^2 = A^3 = 0$, so using (3), (4a) and (40)

$$A_x(0,0) = (4\pi\varepsilon_0 c^2)^{-1}\, v(0, t_0) \int \cdot [x^2 + \gamma^2(y^2+z^2)]^{-1/2}\, \varrho^0(x)\, d^3 x, \quad A_y = A_z = 0, \tag{41}$$

which is also formally the same as for a uniformly moving charge [1].

In the comoving frame K we get from (35), (36) and (39)

$$\phi'^0(x'_p) = (4\pi\varepsilon_0)^{-1} \int \cdot [(x' - x'_p)^2 + \gamma^{-2}(y'^2 + z'^2)]^{-1/2}\, \varrho'^0(x')\, d^3 x', \tag{42}$$

and the vector potential is zero.

4. Discussion

That both procedures give different expressions for the present position integrals is due to the fact that the coaccelerating frame K is holonomic while the Frenet-Serret tetrad is nonholonomic (an excellent discussion of holonomic and nonholonomic frames is given in [29]). Such a difference of course does not exist for uniform motions. One could call H and NH-present position potentials the integrals (14) and (40).

There exists a somewhat similar discrepancy for Maxwell's equations in rotating frames where the holonomic and nonholonomic approaches do not give equivalent equations when one uses the conventional rotation transformation. But recently [23], it has been proved that this discrepancy disappears if a more general rotation transformation is implemented.

So, since many different description of the hyperbolic motion are possible, the question is whether there exists a system of equations such that holonomic and nonholonomic approaches give the same expression for the present position integrals.

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